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On the general problem of stability for impulsive differential equations

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Abstract

Criteria for stability, asymptotical stability and instability of the nontrivial solutions of the impulsive system

$$\frac{dx}{dt} = f(t,x), \quad t \neq \theta_i(x),$$
$$\Delta x|_{t=\theta_i(x)} = I_i(x), \quad i \in \mathcal{N} = \{1,2,\ldots\},$$

where $\Delta x|_{t=\theta} := x(\theta+) - x(\theta)$, $x(\theta+) = \lim_{t\to\theta+} x(t)$ are obtained by Lyapunov's second method. The construction of a reduced system in the neighbourhood of a nontrivial solution is a central auxiliary result of the paper.

Keywords: Stability; Instability; Reduced system; Lyapunov's second method; Impulse effect; Variable moments

1. Introduction

The problem of stability of solutions holds a very significant place in the theory of impulsive differential equations (see [3,8,15] and references cited therein). Milman and Myshkis [12] investigate the stability of the zero solution of differential equations with fixed moments of impulse actions by using the second Lyapunov method. Later, the method was used for differential equations with impulses at variable times, impulsive hybrid systems, for stability criteria in terms of two measures and integro-differential equations [2–6,

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8,10,13,15]. To the best of our knowledge only papers [9,11] deal with the stability of the nontrivial solution of an impulsive system with variable time of the impulse action via Lyapunov direct method. The results of [9] are based on the idea that the surfaces of discontinuity degenerate into vertical planes as time increases infinitely and the assumption that the distance between different solutions does not increase after jumps. In present paper we consider a more general form of the problem without using the conditions mentioned above. It deserves to be emphasised that, apparently, the construction of a reduced system for systems with variable time of impulsive action is done for the first time.

Let $G_x \subset R^n$ be a bounded domain and $G = \{(t, i, x)\colon\ t \in R^+,\ i \in \mathcal{N},\ x \in G_x\}$, where $t_0 \in R$ is fixed, $R^+ = [t_0, +\infty)$, $\mathcal{N} = \{1, 2, 3, \ldots\}$. The main object of the paper is the following system of differential equations with impulse actions on surfaces:

$$\begin{aligned} &\frac{dx}{dt} = f(t, x), \quad t \neq \theta_i(x), \\ &\Delta x|_{t=\theta_i(x)} = I_i(x), \end{aligned} \tag{1}$$

which is considered on the set G and whose solutions are piecewise continuous, with discontinuities of the first kind, left continuous functions.

Let $\|x\|$ denote the Euclidean norm of $x \in R^n$, and $R_+ = [0, \infty)$.

Definition 1.1. A function $h \in C[R_+, R_+]$ is said to belong to class $\mathcal{H}$ if h is strictly increasing and $h(0) = 0$.

Definition 1.2. A function $a \in C[R_+, R_+]$ is said to belong to class $\mathcal{A}$ if $a(0) = 0$ and $a(s) > 0$ for $s > 0$.

We will use the following conditions:

(C1) $f(t, x) : R^+ \times G_x \to R^n$ is a piecewise continuous function with discontinuities of the first kind at boundary points of surfaces $t = \theta_i(x)$, $i \in \mathcal{N}$, where it is left continuous with respect to t, $I_i \in C[G_x, R^n]$, $\theta_i \in C[G_x, R^+]$, $i \in \mathcal{N}$;
(C2) $\sup_G \|f(t, x)\| = M < \infty$;
(C3) There exist a function $\gamma \in \mathcal{H}$ and a number $l > 0$ such that

$$\big\|I_i(x_1) - I_i(x_2)\big\| \leqslant \gamma\big(\|x_1 - x_2\|\big), \qquad \big|\theta_i(x_1) - \theta_i(x_2)\big| \leqslant l\|x_1 - x_2\|$$

for all $i \in \mathcal{N}$, $\{x_1, x_2\} \in G_x$;
(C4) $t_0 < \theta_1(x) < \theta_2(x) < \cdots$, $\theta_i(x) \to \infty$ as $i \to \infty$ for every $x \in G_x$;
(C5) $\theta_i(x + I_i(x)) < \theta_i(x)$ for all $i \in \mathcal{N}$, $x \in G_x$;
(C6) $\theta_{i+1}(x + I_i(x)) > \theta_i(x)$ for all $i \in \mathcal{N}$, $x \in G_x$;
(C7) The existence and uniqueness of solutions of (1) hold.

We should note that the system considered in this paper belongs to a class of systems with impulses at nonfixed moments and, therefore, it needs conditions of the absence of beating [8,15]. We assume that (C5) is valid and $Ml < 1$. Then beating is absent for (1). It is easily seen that conditions (C1) and (C6) on functions $t = \theta_i(x)$ guarantee that a solution of (1) meets every surface of discontinuity if the range of the function $t = \theta_i(x)$, $i \in \mathcal{N}$, is included into the domain of the solution. So we can assume that the following condition of general character is valid:

(C8) A solution $x(t):[t_0,a] \to R^n$ of (1), where $a \in R$, $a > t_0$, or $a = \infty$, intersects any of the surfaces $t = \theta_i(x)$, $i \in \mathcal{N}$, not more than once. And, if $\sup_{G_x} \theta_i(x) < a$, then $x(t)$ intersects $t = \theta_i(x)$ exactly once.

Let $x_0(t)$ be a solution of (1) discontinuous at $t = \tau_i$, $i \in \mathcal{N}$. It follows from assumptions (C3), (C4) and (C8) on the surfaces of discontinuity that the sequence $\tau_i, i = 1, 2, 3, \ldots$, does not have a finite limit point. Thus, if $\{\tau_i\}$, $i \in \mathcal{N}$, is an infinite sequence, then $\tau_i \to \infty$ as $i \to \infty$. Notice that (C4) implies $t_0 \neq \tau_i$ for all $i \in \mathcal{N}$.

A solution $x_0(t)$ of (1) is called continuable to the right if $x(t): R^+ \to R^n$ and $\tau_i \to \infty$ as $i \to \infty$.

Let $T \subseteq R$ be a fixed interval. Define a set $\mathcal{U}_T$ of functions $u: T \to R^n$ which are left continuous with discontinuities of the first kind. Assume that the set of discontinuity points of every function $u \in \mathcal{U}_T$ is not more than countable and does not have a finite limit point in R. Fix $\epsilon \in R$, $\epsilon > 0$.

Definition 1.3. A function $u_2 \in \mathcal{U}_T$ is said to belong to ϵ-neighbourhood of $u_1 \in \mathcal{U}_T$ if: (1) every discontinuity point of $u_2(t)$ lies in ϵ-neighbourhood of a discontinuity point of $u_1(t)$; (2) for all $t \in T$, which are not in ϵ-neighbourhoods of discontinuity points of $u_1(t)$, the inequality $\|u_1(t) - u_2(t)\| < \epsilon$ is valid.

Definition 1.4. Hausdorff's topology, which is built on the basis of all ϵ-neighbourhoods, $0 < \epsilon < \infty$, of all elements $u \in \mathcal{U}_T$, will be called B_T-topology.

Let $x_0(t)$ be a continuable to the right solution of (1).

Definition 1.5. The solution $x_0(t)$ is said to be B-stable in Lyapunov sense if for any positive $\epsilon \in R$ there exists a number $\delta > 0$, such that every solution $x(t)$ of (1) which satisfies $\|x_0(t_0) - x(t_0)\| < \delta$ belongs to ϵ-neighbourhood of $x_0(t)$ in B_{R^+}-topology.

Definition 1.6. A B-stable solution $x_0(t)$ of (1) is called B-asymptotically stable, if there exists a number $\Delta > 0$, such that, if $x(t)$ is a solution of (1) which satisfies an inequality $\|x(t_0) - x_0(t_0)\| < \Delta$, then for any $\epsilon > 0$, a number $\xi > t_0$, exists such that the $x(t)$ lies in ϵ-neighbourhood of $x_0(t)$ in $B_{[\xi,\infty)}$-topology.

Definition 1.7. A solution $x_0(t)$ of (1) is called B-unstable, if either it is not continuable to the right or for some $\epsilon > 0$, and any $\delta > 0$, a solution $x_\delta(t)$ of (1) exists such that $\|x_\delta(t_0) - x(t_0)\| < \delta$ and $x_\delta(t)$ is not in ϵ-neighbourhood of $x_0(t)$ in B_{R^+}-topology.

Remark 1.1. The definitions of stability of nontrivial solutions for systems with nonfixed moments of impulse actions were given in [8,9,15]. The authors of [8,9] name this kind of stability as quasistability. Our definitions [1] are based on the ideas of [7] which were used to define a discontinuous almost periodic function. One can show that Definitions 1.5–1.7 and the definitions of quasistability are equivalent.

Remark 1.2. It is easy to see that by using the language of B-topology one can give definitions of all kinds of stability of solutions as well as of integral manifolds. If the moments of impulse effects are fixed then the definitions of stability coincide with the classic definitions [14].

Fix $h^0 \in R$, $h^0 > 0$, denote $G^0 = \{(t,x) \mid t \in R^+,\ \|x\| < h^0\}$, and let $V(t,x) \in C^1[G^0, R_+]$ and $V(t,0) = 0$ for all $t \in R^+$.

Definition 1.8. The function $V(t,x)$ is said to be positive definite on G^0 if there exists $a \in \mathcal{H}$ such that $V(t,x) \geqslant a(\|x\|)$ for all $(t,x) \in G^0$; it is called positive semidefinite on G^0 if $V(t,x) \geqslant 0$ for all $(t,x) \in G^0$. The function $V(t,x)$ is called negative definite (negative semidefinite) on G^0 if $-V(t,x)$ is positive definite (positive semidefinite) on G^0.

We will use the notation

$$\dot{V}_f(t,x) = \frac{\partial V(t,x)}{\partial x} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x),$$

where $x = (x_1, \dots, x_n)$ and $f(t,x) = (f_1, \dots, f_n)$.

2. *B*-reduced system

Let $x_0(t)$ be a continuable to the right solution of (1), and τ_i, $i \in \mathcal{N}$, be discontinuity points of $x_0(t)$, i.e., $\tau_i = \theta_i(x_0(\tau_i))$, $i \in \mathcal{N}$. Assume that $x(t): R^+ \to R^n$ is another solution of (1) and γ_i, $i \in \mathcal{N}$, are discontinuity points of $x(t)$, $\gamma_i = \theta_i(x(\gamma_i))$, $i \in \mathcal{N}$. One can show that the difference $z(t) = x(t) - x_0(t)$ satisfies the following system of equations:

$$\begin{aligned}
&\frac{dz}{dt} = f\big(t, x_0(t) + z\big) - f\big(t, x_0(t)\big), \quad t \neq \tau_i,\ t \neq \gamma_i,\\
&\Delta z|_{t=\tau_i} = -I_i\big(x_0(\tau_i)\big), \quad \Delta z|_{t=\gamma_i} = I_i\big(x(\gamma_i)\big),\\
&\Delta z|_{t=\gamma_i=\tau_i} = I_i\big(x(\gamma_i)\big) - I_i\big(x_0(\tau_i)\big), \quad i \in \mathcal{N}.
\end{aligned} \tag{2}$$

As the points $t = \gamma_i$, $i \in \mathcal{N}$, depend on a solution $x(t)$, it is not easy to investigate stability of the zero solution of (2). So we suggest to use another way of investigation as follows.

Fix $i \in \mathcal{N}$, $x \in G_x$, and let $\xi(t)$ be a solution of the system

$$\frac{dx}{dt} = f(t,x) \tag{3}$$

with initial condition $\xi(\tau_i) = x$. Let $t = \zeta_i$ be a meeting moment such that $\zeta_i = \theta_i(\xi(\zeta_i))$.

Further we shall accept $\widehat{[\tau_i, \zeta_i]}$ as well as $\widehat{(\tau_i, \zeta_i]}$ as oriented intervals, that is

$$\widehat{[\tau_i, \zeta_i]} = \begin{cases} [\zeta_i, \tau_i] & \text{if } \zeta_i \leqslant \tau_i,\\ [\tau_i, \zeta_i] & \text{otherwise.} \end{cases}$$

Assuming that solutions $\xi(t)$ and $\xi_1(t)$ of (3), $\xi_1(\zeta_i) = \xi(\zeta_i) + I_1(\xi(\zeta_i))$ are given on interval $\widehat{[\tau_i, \zeta_i]}$ we construct a map $\Phi_i : \{\tau_i\} \times G_x \to \{\tau_i\} \times R^n$ such that

$$\Phi_i(x) = \int_{\tau_i}^{\zeta_i} f\bigl(s, \xi(s)\bigr)\,ds + I_i\left(x + \int_{\tau_i}^{\zeta_i} f\bigl(s, \xi(s)\bigr)\,ds\right) + \int_{\zeta_i}^{\tau_i} f\bigl(s, \xi_1(s)\bigr)\,ds.$$

Denote

$$h_0 = \inf_{\substack{t \in R^+ \\ x \in \partial G_x}} \bigl\|x_0(t) - x\bigr\|, \quad h_0 > 0,$$

where ∂G_x is the boundary of the domain G_x and introduce new functions

$$\kappa(s) = \frac{1+3lM}{1-lM}s + \gamma\left(\frac{s}{1-lM}\right), \qquad \mu(s) = \frac{2lM}{1-lM}s + \gamma\left(\frac{lMs}{1-lM}\right),$$
$$\pi(s) = \frac{2lM}{1-lM}s + \gamma\left(\frac{s}{1-lM}\right),$$

where $s \geqslant 0$. Obviously, $\{\kappa, \mu, \pi\} \subset \mathcal{H}$. Let $h \in R$ be such that $h > 0$ and $\kappa(h) = h_0$, and

$$G_i = \bigl\{x \in G_x \mid \bigl\|x - x_0(\tau_i)\bigr\| < h\bigr\}, \qquad G_i^+ = \bigl\{x \in G_x \mid \bigl\|x - x_0(\tau_i+)\bigr\| < h\bigr\},$$
$$G^i = \left[\tau_i - \frac{lh}{1-lM}, \tau_i + \frac{lh}{1-lM}\right] \times G_i \cup G_i^+,$$
$$G_h = \bigl\{(t, x) \in G \mid t \in R^+,\ \bigl\|x - x_0(t)\bigr\| < h\bigr\} \cup \left(\bigcup_{i \in \mathcal{N}} G^i\right),$$

where $\times$ is the sign of the Cartesian product.

Let us consider the system

$$\begin{aligned} &\frac{dy}{dt} = f(t, y), \quad t \neq \tau_i, \\ &\Delta y|_{t=\tau_i} = \Phi_i(y). \end{aligned} \tag{4}$$

Definition 2.1. Systems (1) and (4) are said to be B-equivalent on $G_h \times \mathcal{N}$, if for every solution $x(t) : [t_0, a) \to R^n$, $a \in R^+ (a = \infty)$, $(t, x(t)) \in G_h$, of (1), there exists a solution $y(t)$, $y(t_0) = x(t_0)$, of (4), such that

$$x(t) = y(t), \quad t \in [t_0, a) \Big\backslash \bigcup_{i \in \mathcal{N}} \widehat{(\tau_i, \zeta_i]}. \tag{5}$$

Specifically,

$$x(\tau_i) = y(\tau_i), \quad x(\zeta_i+) = y(\zeta_i) \quad \text{if } \tau_i \leqslant \zeta_i, \tag{6}$$
$$x(\tau_i) = y(\tau_i+), \quad x(\zeta_i) = y(\zeta_i) \quad \text{if } \tau_i > \zeta_i. \tag{7}$$

And, conversely, for every solution $y(t) : [t_0, a) \to R^n$, $a \in R^+$ $(a = \infty)$, of (4), $(t, y(t)) \in G_h$, there exists a solution $x(t)$, $x(t_0) = y(t_0)$, of (1), which satisfies (5)–(7).

Lemma 2.1. *The following assertions are valid*:

(i) $\Phi_i(x_0(\tau_i)) = I_i(x_0(\tau_i))$, $i \in \mathcal{N}$;
(ii) $\Phi_i : G_i \to R^n$, $i \in \mathcal{N}$;
(iii) $\|\Phi_i(x) - I_i(x)\| \leqslant \mu(\|x - x_0(\tau_i)\|)$ *for all* $i \in \mathcal{N}$, $x \in G_i$;
(iv) $\|\Phi_i(x) - I_i(x_0(\tau_i))\| \leqslant \pi(\|x_0(\tau_i) - x\|)$ *for all* $i \in \mathcal{N}$, $x \in G_i$.

Proof. Assertion (i) immediately follows from definition of $\Phi_i(x)$. We have, for fixed $x \in G_x$, that $\xi(t) = x + \int_{\tau_i}^t f(s, \xi(s))\, ds$, $t \in \widehat{[\tau_i, \zeta_i]}$, and $\xi_1(t) = \xi(\zeta_i) + I_i(\xi(\zeta_i)) + \int_{\zeta_i}^t f(s, \xi_1(s))\, ds$, $t \in \widehat{[\tau_i, \zeta_i]}$. Then

$$\bigl\|\xi(t) - x_0(t)\bigr\| \leqslant \bigl\|x - x_0(\tau_i)\bigr\| + M|\zeta_i - \tau_i| \tag{8}$$

for all $t \in \widehat{[\tau_i, \zeta_i]}$. Hence, $|\tau_i - \zeta_i| = |\theta_i(x_0(\tau_i)) - \theta_i(\xi(\zeta_i))| \leqslant l(\|x_0(\tau_i) - x\| + M|\tau_i - \zeta_i|)$ and

$$|\tau_i - \zeta_i| \leqslant l \frac{\|x - x_0(\tau_i)\|}{1 - lM}. \tag{9}$$

From (8) and (9) it follows that

$$\bigl\|\xi(t) - x_0(\tau_i)\bigr\| \leqslant \frac{\|x_0(\tau_i) - x\|}{1 - lM} \quad \text{if } t \in \widehat{[\tau_i, \zeta_i]} \tag{10}$$

and

$$\begin{aligned} \bigl\|\xi_1(t) - x_0(\tau_i+)\bigr\| &= \left\| \xi(\zeta_i) + I_i\bigl(\xi(\zeta)\bigr) + \int_{\zeta_i}^t f\bigl(s, \xi_1(s)\bigr)\, ds - x_0(\tau_i) - I_i\bigl(x_0(\tau_i)\bigr) \right\| \\ &\leqslant \kappa\bigl(\bigl\|x_0(\tau_i) - x\bigr\|\bigr) \end{aligned} \tag{11}$$

for all $t \in \widehat{[\tau_i, \zeta_i]}$. The equality $\Phi_i(x) = \xi_1(\tau_i) - x$ and (10) and (11) imply that $\Phi_i(x)$ is defined on G_i and assertions (iii) and (iv) are true. The lemma is proved. □

Theorem 2.1. *Systems* (1) *and* (4) *are B-equivalent on $G_h \times \mathcal{N}$. The function $x = x_0(t)$ is a solution of systems* (1) *and* (4) *simultaneously.*

Proof. Let $x(t)$ and $y(t)$ be solutions of (1) and (4), respectively, such that $x(t_0) = y(t_0)$. Without loss of generality we can assume that $[t_0, \tau_1]$ is an interval of continuity of solutions $x(t)$ and $y(t)$ and, hence, $(t, x(t)) \in G_h$ for all $t \in [t_0, \tau_1]$. It is obvious that $\zeta_1 \geqslant \tau_1$. Since the case $\zeta_1 = \tau_1$ is trivial, we shall consider only the case $\zeta_1 > \tau_1$. If $(t, x(t)) \in G_h$, $t \in [\tau_1, \zeta_1]$, then

$$\begin{aligned} \bigl\|y(t) - x_0(t)\bigr\| = \Bigl\| & y(\tau_1) + \Phi_1\bigl(y(\tau_1)\bigr) + \int_{\tau_1}^t f\bigl(s, y(s)\bigr)\, ds \\ & - x_0(\tau_1) - I_1\bigl(x_0(\tau_1)\bigr) - \int_{\tau_1}^t f\bigl(s, x_0(s)\bigr)\, ds \Bigr\| \end{aligned}$$

$$< h + \pi(h) + \frac{2lMh}{1 - lM} = \kappa(h) = h_0.$$

If it is known that $(t, x) \in G^i$ for all $t \in [\tau_1, \zeta_1]$, then similarly to (10) one can show that

$$\begin{aligned}\|x(t) - x_0(\tau_1)\| &= \left\| y(\tau_1) + \int_{\tau_1}^{t} f\bigl(s, x(s)\bigr)\,ds - x_0(\tau_1) \right\| \\ &< \frac{h}{1 - lM} < \kappa(h) = h_0.\end{aligned}$$

Moreover, $x(\zeta_1+) = y(\zeta_1)$; in view of the definition of $\Phi_1(x)$. Thus, (C7) implies that $x(t) = y(t)$ if t is a continuity point and $t > \zeta_1$. Then one can verify that conditions of Definition 2.1 are valid for all $t \in [t_0, a)$. The assertion about $x_0(t)$ is trivial. The theorem is proved. □

Let $x(t)$ be a solution of (1) and an integral curve of $x(t)$ belongs to G_h. Let $y(t)$ be a solution of (1), $y(t_0) = x(t_0)$, which corresponds to $x(t)$ by B-equivalence in $G_h \times \mathcal{N}$. If $u = y(t) - x_0(t)$ then $u(t)$ satisfies the following system:

$$\begin{aligned}&\frac{du}{dt} = F(t, u), \quad t \neq \tau_i, \\ &\Delta u|_{t=\tau_i} = J_i(u) + W_i(u),\end{aligned} \tag{12}$$

where

$$\begin{aligned}&F(t, u) = f\bigl(t, x_0(t) + u\bigr) - f\bigl(t, x_0(t)\bigr), \qquad J_i(u) = I_i\bigl(x_0(\tau_i) + u\bigr) - I_i\bigl(x_0(\tau_i)\bigr), \\ &W_i(u) = \Phi_i\bigl(x_0(\tau_i) + u\bigr) - I_i\bigl(x_0(\tau_i) + u\bigr).\end{aligned} \tag{13}$$

Definition 2.2. System (12) is said to be a B-reduced system for (1) in the vicinity of $x_0(t)$.

Theorem 2.2. $\|W_i(u)\| \leqslant \mu(\|u\|)$ *if* $\|u\| < h$.

Proof. The validity of the theorem follows immediately from condition (iii) of Lemma 2.1 and the last equality in (13). □

Remark 2.1. It is obvious that W_i are functionals of solutions of (3) and, hence, they cannot be defined explicitly as well as F and J_i. But our intention is to use a qualitative property of W_i which is given by Theorem 2.2.

3. Stability

In this section we will formulate and prove the theorems of stability and unstability. They are analogues of Lyapunov and Chetaev theorems [8,14,15].

Lemma 3.1. *Suppose that on* G^0 *the following conditions are fulfilled*:

(I_1) *$V(t,x)$ is positive definite*;
(I_2) *$\dot V_F(t,x)$ is negative semidefinite*;
(I_3) *There exists a function $\alpha \in \mathcal{A}$ such that $V(\tau_i,x) - V(\tau_i, x + J_i(x)) \leqslant -\alpha(\|x\|)$*;
(I_4) *There exists a function $\beta \in \mathcal{A}$ such that $\|\partial V/\partial x\| \leqslant \beta(\|x\|)$*;
(I_5) *$\beta(s)\mu(s) - \alpha(s) \leqslant 0$ if $s > 0$ is sufficiently small.*

Then the trivial solution of (12) *is stable.*

Proof. The conditions imply that

$$\begin{aligned}
&V\big(\tau_i, x + J_i(x) + W_i(x)\big) - V(\tau_i, x)\\
&\quad = V\big(\tau_i, x + J_i(x)\big) - V(\tau_i, x) - V\big(\tau, x + J_i(x)\big) + V\big(\tau_i, x + J_i(x) + W_i(x)\big)\\
&\quad \leqslant \beta\big(\|x\|\big)\mu\big(\|x\|\big) - \alpha\big(\|x\|\big) \leqslant 0
\end{aligned}$$

if $\|x\|$ is sufficiently small. Thus, all the assumptions of Theorem 47 of [15] are fulfilled and the proof is complete. □

Theorem 3.1. *Suppose that conditions* (C1)–(C8) *and* (I_1)–(I_5) *are fulfilled. Then the solution $x_0(t)$ of* (1) *is B-stable.*

Proof. Fix $\epsilon > 0$ and denote $\epsilon_1 = \epsilon \min(1, (1 - lM)/l)$. Since (12) is the reduced system [15] of (4), then by above lemma the solution $x_0(t)$ of (4) is stable, i.e., there exists $\delta > 0$, such that if $y(t)$, $\|y(t_0) - x_0(t_0)\| < \delta$, is a solution of (4), then $\|y(t) - x_0(t)\| < \epsilon_1$, $t \in R^+$. Let $x(t)$, $x(t_0) = y(t_0)$, be a solution of (1). The B-equivalence implies that

$$\big\|x(t) - x_0(t)\big\| < \epsilon_1, \quad t \notin (\tau_i, \zeta_i],\ i \in \mathcal{N}, \tag{14}$$

where ζ_i, $i \in \mathcal{N}$, are the discontinuity points of $x(t)$. Assume without any loss of generality that $\zeta_i \geqslant \tau_i$. We have that $\zeta_i - \tau_i = \theta_i(x(\zeta_i)) - \theta_i(x(\tau_i)) < l(\|x(\zeta_i) - x_0(\tau_i)\|) \leqslant l(\|x(\tau_i) - x_0(\tau_i)\| + M(\zeta_i - \tau_i))$ and

$$\zeta_i - \tau_i < \frac{l\epsilon_1}{1 - lM} = \epsilon. \tag{15}$$

The proof of the theorem follows from (14) and (15). □

Lemma 3.2. *Suppose that conditions* (I_1)–(I_4) *are valid and, moreover, the following assumption is fulfilled*:

(I_6) *There exists a function $\psi \in \mathcal{A}$ such that $\beta(\|x\|)\mu(\|x\|) - \alpha(\|x\|) \leqslant -\psi(V(\tau_i,x))$ for sufficiently small $\|x\|$.*

Then the zero solution of (12) *is asymptotically stable.*

Proof. Similarly to the proof of Lemma 3.1 one can find that $V(\tau_i, x + J_i(x) + W_i(x)) - V(\tau_i, x) \leqslant -\psi(V(\tau_i, x))$ for sufficiently small $\|x\|$ and, hence, all the conditions of Theorem 47 of [15] for the asymptotic stability are fulfilled. □

Theorem 3.2. *Suppose that conditions* (C1)–(C8) *and* (I_1)–(I_4), (I_6) *are fulfilled. Then the solution* $x_0(t)$ *of* (1) *is asymptotically stable.*

Proof. Since all conditions of Theorem 3.1 are valid, then the solution $x_0(t)$ of (1), is B-stable. Moreover, (12) is the reduced system for (4) and all the conditions of Lemma 3.2 are fulfilled. Hence, $x_0(t)$ is an asymptotically stable solution of (4). That is, for $\epsilon_1 > 0$ there exists $\delta_1 > 0$, such that if $y(t)$, $\|y(t_0) - x_0(t_0)\| < \delta_1$, is a solution of (4), then there exists $\xi \in R$, $\xi = \xi(y, \epsilon)$, $\xi > t_0$, such that $\|y(t) - x_0(t)\| < \epsilon_1$, $t \geqslant \xi$. Let $x(t)$, $x(t_0) = y(t_0)$, be a solution of (1). Then similarly to the proof of Theorem 3.1 one can show that (14) and (15) are valid for $t \geqslant \xi$. That is, $x(t)$ is in ϵ-neighbourhood, $\epsilon_1 = \epsilon \min(1, (1 - lM)/l)$, of $x_0(t)$ in the $B_{[\xi,\infty)}$-topology. The theorem is proved. □

We shall formulate the following Theorems 3.3 and 3.4 without proof. They can be verified by the same techniques as Theorems 3.1 and 3.2 using Theorems 48 and 49 of [15].

Theorem 3.3. *Suppose that conditions* (C1)–(C8), (I_1), (I_4) *are fulfilled. Moreover, the following assumptions are valid*:

(V_0) *There exists a number* $\theta > 0$: $\inf_i(\tau_{i+1} - \tau_i) = \theta$;
(V_1) *There exists a function* $\phi \in \mathcal{A}$ *such that* $\dot{V}_F(t, x) \leqslant -\phi(V(t, x))$ *for all* $(t, x) \in G^0$;
(V_2) *There exists a function* $\psi \in \mathcal{A}$ *such that* $V(\tau_i, x + J_i(x)) \leqslant \psi(V(\tau_i, x)) - \beta(\|x\|) \times \mu(\|x\|)$ *for all* $i \in \mathcal{N}$ *and sufficiently small* $\|x\|$;
(V_3) *There exist numbers* $a_0 > 0$ *and* $\nu \geqslant 0$ *such that* $\int_a^{\psi(a)} (1/\phi(s))\, ds \leqslant \theta - \nu$ *for all* $a \in (0, a_0]$.

Then the solution $x_0(t)$ *of* (1) *is* B*-stable if* $\nu = 0$ *and it is* B*-asymptotically stable if* $\nu > 0$.

Theorem 3.4. *Let conditions* (C1)–(C8), (I_1), (I_4), (V_2) *be valid and, moreover, the following assumptions be fulfilled*:

(V_4) *There exists* $\theta_1 \in R$, $\theta_1 < \infty$, *such that* $\sup_i(\tau_{i+1} - \tau_i) = \theta_1$;
(V_5) *There exists a function* $\phi \in \mathcal{A}$ *such that* $\dot{V}_F(t, x) \leqslant \phi(V(t, x))$ *for all* $(t, x) \in G^0$;
(V_6) *There exist numbers* $a_0 > 0$ *and* $\nu \geqslant 0$ *such that* $\int_{\psi(a)}^{a} (1/\phi(s))\, ds \geqslant \theta_1 + \nu$ *for all* $a \in (0, a_0]$.

Then the solution $x_0(t)$ *of* (1) *is* B*-stable if* $\nu = 0$ *and it is* B*-asymptotically stable if* $\nu > 0$.

Let us make an additional assumption that

(C9) $\inf_i[\inf_{G_x} \theta_{i+1}(x) - \sup_{G_x} \theta_i(x)] = q > 0$.

Lemma 3.3. *If conditions* (C1)–(C9) *are fulfilled, then* B*-stability of the solution* $x_0(t)$ *of* (1) *implies that it is a stable solution of* (4).

Proof. Fix $\epsilon \in R$, $0 < \epsilon < q/(1+2M)$, and denote $\epsilon_1 = \epsilon(1+2M)^{-1}$, $T = \{t \in R^+ \mid |t - \tau_l| > \epsilon_1,\ i \in \mathcal{N}\}$. There exists $\delta = \delta(\epsilon_1) > 0$ such that a solution $x(t)$, $\|x(t_0) - x_0(t_0)\| < \delta$, of (1) satisfies the inequalities $|\tau_i - \zeta_i| < \epsilon_1$, $i \in \mathcal{N}$, and $\|x(t) - x_0(t)\| < \epsilon_1$, $t \in T$, where ζ_i, $i \in \mathcal{N}$, are the points of discontinuity of $x(t)$. Let $y(t)$, $y(t_0) = x(t_0)$, be a solution of (4). Since $T \cap \widehat{[\tau_i, \zeta_i]} = \emptyset$, $i \in \mathcal{N}$, it is true that $\|y(t) - x_0(t)\| < \epsilon_1 < \epsilon$, $t \in T$.

Let $t \notin T$. Then the following cases are possible: (a) $t \notin (\tau_i, \zeta_i]$, $i \in \mathcal{N}$; (b) there exists $j \in \mathcal{N}$, such that $t \in \widehat{(\tau_j, \zeta_j]}$. We shall consider these cases in turn.

(a) Let us assume that $\tau_j \leqslant \zeta_j < t$, $[\tau_j, t] \cap T = \emptyset$ (other possibilities can be considered similarly). Denote

$$t^* = \tau_j + \epsilon_1. \tag{16}$$

Using (C9) one can verify that

$$\left\|x(t^*) - x_0(t^*)\right\| \leqslant \epsilon_1, \quad x(t^*) = y(t^*). \tag{17}$$

Then

$$\begin{aligned} \left\|y(t) - x_0(t)\right\| &= \left\| y(t^*) + \int_{t^*}^{t} f\big(s, y(s)\big)\, ds - x_0(t^*) - \int_{t^*}^{t} f\big(s, x_0(s)\big)\, ds \right\| \\ &< \epsilon_1 + 2M\epsilon_1 = \epsilon. \end{aligned} \tag{18}$$

(b) Now let $\tau_j < t \leqslant \zeta_j$ (similarly, one can investigate the case $\zeta_j < t \leqslant \tau_j$). Defining t^* again by (16) and (17), one can see that (18) is valid and the lemma follows. □

Lemma 3.4. *Suppose that condition* (I_4) *is fulfilled and the following assumptions are valid*:

(W_1) *The intersection of the domain* $\mathcal{P} = \{(t, x) \in G^0 \mid V(t, x) > 0\}$, *and the plane* $t =$ const *is a nonempty open set adherent to the origin for any* $t \in R^+$;
(W_2) $V(t, x)$ *is bounded on* $\mathcal{P}$;
(W_3) $\dot{V}_F(t, x)$ *is positive semidefinite on* $\mathcal{P}$;
(W_4) *There exists a function* $\psi \in \mathcal{A}$ *such that* $V(\tau_i, x + J_i(x)) - V(\tau_i, x) \geqslant \psi(V(\tau_i, x)) + \beta(\|x\|)\mu(\|x\|)$ *for all* $i \in \mathcal{N}$, $x \in \mathcal{P}$.

Then, the zero solution of (12) *is unstable.*

The proof follows from Theorem 50 of [15] similarly to the proof of Lemma 3.2.

Theorem 3.5. *Suppose that conditions* (C1)–(C9) *and* (W_1)–(W_4) *are fulfilled. Then the solution* $x_0(t)$ *of* (1) *is* B*-unstable.*

Proof. The proof follows immediately from Lemmas 3.3 and 3.4. □

4. Examples

Example 4.1. Consider the following system with nonfixed moments of impulse actions:

$$\begin{aligned}
&\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = (-1)^{i+1}\sin x_1, \quad t \neq \theta_i(x),\\
&\Delta x_1|_{t=\theta_i(x)} = ax_1 + bx_2 + \pi\left[(-1)^i - a\frac{1-(-1)^i}{2}\right],\\
&\Delta x_2|_{t=\theta_i(x)} = -bx_1 + ax_2 + b\pi\frac{1-(-1)^i}{2},
\end{aligned} \tag{19}$$

where $x = (x_1, x_2) \in R^2$ and a and b are constants such that

$$(a+1)^2 + b^2 < 1 \tag{20}$$

and

$$\theta_i(x) = i\theta - (-1)^i\lambda x_1 + \sigma x_2 - \frac{\pi\lambda}{2}\bigl(1-(-1)^i\bigr), \quad i \in \mathcal{N}, \tag{21}$$

where λ, θ, σ are constants such that $\theta > 0$, $\lambda > 0$ and

$$\theta > \lambda\pi. \tag{22}$$

Denote $h = 1 - (a+1)^2 - b^2$ and

$$G = \left\{(t, i, x) \mid t \geqslant t_0,\ i \in \mathcal{N},\ \|x\| < \frac{\pi}{h}\right\},$$

where $t_0 \in R$, $0 < t_0 < \theta$, is fixed. We assume that λ and $|\sigma|$ are sufficiently small such that surfaces $t = \theta_i(x)$, $i \in \mathcal{N}$, do not intersect in G. Thus, the set G is a partition of sets G_i, $i \in \mathcal{N}$, where G_1 is a part of G which is between the surfaces $t = t_0$ and $t = \theta_1(x)$, and G_i is a part of G which is between surfaces $t = \theta_{i-1}(x)$ and $t = \theta_i(x)$, and the surface $t = \theta_i(x)$ is included in G_i. In G_i, $i = 2k-1$, $k \in \mathcal{N}$, system (19) has a form

$$\begin{aligned}
&\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \sin x_1, \quad t \neq \theta_i(x),\\
&\Delta x_1|_{t=\theta_i(x)} = ax_1 + bx_2 - (a+1)\pi,\\
&\Delta x_2|_{t=\theta_i(x)} = -bx_1 + ax_2 + b\pi,
\end{aligned} \tag{23}$$

where $\theta_i(x) = i\theta + \lambda(x_1 - \pi) + \sigma x_2$.

If $i = 2k$, $i \in \mathcal{N}$, then system (19) has another form in G_i,

$$\begin{aligned}
&\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -\sin x_1, \quad t \neq \theta_i(x),\\
&\Delta x_1|_{t=\theta_i(x)} = ax_1 + bx_2 + \pi,\\
&\Delta x_2|_{t=\theta_i(x)} = -bx_1 + ax_2,
\end{aligned} \tag{24}$$

where $\theta_i(x) = i\theta - \lambda x_1 + \sigma x_2$. One can verify that a piecewise constant function $\xi(t) = (\phi(t), \psi(t))$, where

$$\phi(t) = \begin{cases} \pi & \text{if } t \in [t_0, \theta] \bigcup_{i=2k}(i\theta, (i+1)\theta],\\ 0 & \text{if } t \in \bigcup_{i=2k-1}(i\theta, (i+1)\theta], \end{cases}$$

and $\psi(t) = 0$, for all $t \in R^+$, is a solution of (19) ((23) + (24)) and $\tau_i = i\theta$, $i \in \mathcal{N}$, are discontinuity points of $\xi(t)$. Notice that $\xi(t)$ intersects every surface of discontinuity exactly one time. Indeed, if $i = 2k - 1$, $k \in \mathcal{N}$, then

$$\theta\big(\xi(\tau_i) + \Delta\xi(\tau_i)\big) = i\theta - \lambda\pi < i\theta = \theta_i\big(\xi(\tau_i)\big),$$
$$\theta\big(\xi(\tau_i)\big) = i\theta < (i+1)\theta - \lambda\pi = \theta_{i+1}\big(\xi(\theta_i) + \Delta\xi(\tau_i)\big). \tag{25}$$

The last inequality is true in view of (22). And if $i = 2k$, $k \in \mathcal{N}$, then

$$\theta\big(\xi(\tau_i) + \Delta\xi(\tau_i)\big) = i\theta - \lambda\pi < i\theta = \theta_i\big(\xi(\tau_i)\big),$$
$$\theta\big(\xi(\tau_i)\big) = i\theta < (i+1)\theta = \theta_{i+1}\big(\xi(\theta_i) + \Delta\xi(\tau_i)\big). \tag{26}$$

Thus, (25) and (26) imply that conditions (C5) and (C6) are fulfilled. Denote by C^0 the union of η-neighbourhoods, $\eta \in R$, $\eta > 0$, of the points $(0;0)$ and $(\pi;0)$ in R^2, $M = \sup_{C^0}(x_2^2 + \sin^2(x_1))$, $l = (1/2)\max(\lambda, |\sigma|)$, $l_1 = (1/2)\max(|a|, |b|)$. One can choose η so that $lM < 1$. Moreover, inequalities (25) and (26) are valid uniformly with respect to $i \in \mathcal{N}$ and the functions $t = \theta_i(x)$ are uniformly continuous for all $i \in \mathcal{N}$. Thus, in view of continuity of the functions in the impulse part of (19) and of functions $t = \theta_i(x)$, one can conclude that there is a neighbourhood G^0 of $\xi(t)$ in B_{R^+}-topology such that every solution $x(t) \in G^0$, $t \in R^+$, intersects exactly one time every surface of discontinuity. Denote

$$V(x) = 1 - \cos x_1 + \frac{x_2^2}{2}, \qquad F_1(t,x) = x_2, \qquad F_2(t,x) = -\sin x_1,$$
$$J_i^{(1)}(x) = ax_1 + bx_2, \qquad J_i^{(2)}(x) = -bx_1 + ax_2.$$

The system

$$\frac{dx_1}{dt} = F_1(t,x), \quad \frac{dx_2}{dt} = F_2(t,x), \quad t \neq \theta_i(x),$$
$$\Delta x_1|_{t=\tau_i} = J_i^{(1)}(x), \qquad \Delta x_2|_{t=\tau_i} = J_i^{(2)}(x), \tag{27}$$

is a B-reduced system of Eq. (19) in a neighbourhood of $\xi(t)$. It is not difficult to verify that

$$\dot{V}_F(x) \leqslant 0 \tag{28}$$

and

$$V(x) - V\big(x + J_i(x)\big) = \left[\frac{h}{2} + \kappa(x)\right]\|x\|^2,$$

where $\kappa(x) \to 0$ as $\|x\| \to 0$.

Fix a number $\epsilon \in R$, $0 < \epsilon < h/2$, and denote $\alpha(s) = (h/2 + \epsilon)s^2$. Then

$$V(x) - V\big(x + J_i(x)\big) \leqslant \alpha\big(\|x\|\big) \tag{29}$$

if $\|x\|$ is sufficiently small. Moreover,

$$\left\|\frac{\partial V}{\partial x}\right\| \leqslant \beta\big(\|x\|\big) \tag{30}$$

for $\beta(s) = 2s$ if $\|x\|$ is sufficiently small. Thus,

$$\alpha\bigl(\|x\|\bigr) - \beta\bigl(\|x\|\bigr)\mu\bigl(\|x\|\bigr) = \left[\frac{h}{2} + \epsilon^0 - \frac{2lM(2+l_1)}{1-lM}\right]\|x\|^2,$$

where $\mu(s) = 2lM(2+l_1)s/(1-lM)$. Let

$$\frac{h}{2} + \epsilon - \frac{2lM(2+l_1)}{1-lM} > 0.$$

One can verify that $V(x) = \|x\|^2/2 + \zeta(x)$, where the series for $\zeta(x)$ starts with not less than the third degree. Hence, there exists a function $\Psi \in \mathcal{A}$ such that $\|x\|^2 = \Psi(V(x))$. Denoting

$$\psi\bigl(V(x)\bigr) = \left[\frac{h}{2} + \epsilon - \frac{2lM(2+l_1)}{1-lM}\right]\Psi\bigl(V(x)\bigr),$$

we have that all conditions of Theorem 3.2 are valid and $\xi(t)$ is a B-asymptotically stable solution of (19). As ϵ is arbitrarily small one can conclude that $\xi(t)$ is B-asymptotically stable if $lM(2+l_1)/(1-lM) < h/4$.

Example 4.2. Consider the following system:

$$\begin{aligned}
&\frac{dx_1}{dt} = x_2 + (-1)^i \sin^3(x_1), \quad \frac{dx_2}{dt} = (-1)^{i+1}\sin x_1 + x_2^3, \quad t \neq \theta_i(x),\\
&\Delta x_1|_{t=\theta_i(x)} = ax_1 + bx_2 + \pi\left[(-1)^i - a\frac{1-(-1)^i}{2}\right],\\
&\Delta x_2|_{t=\theta_i(x)} = -bx_1 + ax_2 + \frac{\pi b}{2}\bigl[1-(-1)^i\bigr].
\end{aligned} \tag{31}$$

We will stick to system (31) from the previous example: the set G, the surfaces of discontinuity (21), relations (20) and (22) and constants M, l, l_1. Also assume that $lM < 1$. One can show that the function $\xi(t)$ from Example 4.1 is also a solution of (31) and this solution satisfies relations (25) and (26). The reduced system for (31) in a neighbourhood of $\xi(t)$ has the form (27), where

$$\begin{aligned}
&F_1(t,x) = x_2 + \sin^3(x_1), \qquad F_2(t,x) = -\sin(x_1) + x_2^3,\\
&J_i^{(1)}(x) = ax_1 + bx_2, \qquad J_i^{(2)}(x) = -bx_1 + ax_2.
\end{aligned}$$

Thus, we have that every solution of (31) intersects every surface of discontinuity exactly once if it belongs to a sufficiently small neighbourhood of $\xi(t)$ in B_{R^+}-topology.

Take again as a Lyapunov function the expression $V(x) = 1 - \cos x_2 + x_2^2/2$. It is easily seen that $\dot V_F(x) = \sin^4(x_1) + x_2^4 \leqslant V^2(x)$ if $\|x\|$ is sufficiently small. Moreover,

$$V\bigl(x + J_i(x)\bigr) - V(x) = \left[-\frac{h}{2} + \kappa(x)\right]\|x\|^2, \tag{32}$$

where $\kappa(x) \to 0$ as $\|x\| \to 0$. We will show that the inequality

$$\frac{lM(2+l_1)}{1-lM} < \frac{h}{2}$$

implies that $\xi(t)$ is B-asymptotically stable. Denote

$$h_1 = \frac{h}{2} - \frac{lM(2+l_1)}{1-lM}.$$

Let $\|x\|$ be sufficiently small such that $|\kappa(x)| < \epsilon$, where ϵ, $0 < \epsilon < h_1$, is fixed. Then (32) and inequality $2V(x) \leqslant \|x\|^2$ imply that

$$\begin{aligned} V\big(x + J_i(x)\big) &= V(x) - \left(\frac{h}{2} - \kappa(x)\right)\|x\|^2 \\ &\leqslant V(x)\big[1 - 2(h_1 - \epsilon)\big] - \beta\big(\|x\|\big)\mu\big(\|x\|\big), \end{aligned}$$

where $\beta(s)$ and $\mu(s)$ are as defined in the previous example. If we denote $\psi(s) = [1 - 2(h_1 - \epsilon)]s$, then the condition ($V_2$) is valid and

$$\int\limits_{\psi(a)}^{a} \frac{ds}{\phi(s)} \equiv \int\limits_{(1-2(h_1-\epsilon))a}^{a} \frac{ds}{s^2} = \frac{2(h_1 - \epsilon)}{a(1 + 2(\epsilon - h_1))}.$$

Since the inequality

$$\frac{2(h_1 - \epsilon)}{a(1 + 2(\epsilon - h_1))} \geqslant \theta_1 + \nu, \quad \theta_1 = \theta,$$

is true if a is sufficiently small, we can conclude that condition (V_5) is also valid and $\xi(t)$ is a B-asymptotically stable solution of (31) by Theorem 3.4.

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References

[1] M.U. Akhmetov, N.A. Perestyuk, A comparison method for differential equations with impulse actions, Differential Equations 26 (1990) 1079–1086.
[2] M.U. Akhmetov, A. Zafer, Stability of the zero solution of impulsive differential equations by the Lyapunov second method, J. Math. Anal. Appl. 248 (2000) 69–82.
[3] D.D. Bainov, P.S. Simeonov, Systems with Impulse Effect. Stability, Theory and Applications, Ellis Horwood/Halsted/Wiley, Chichester/New York, 1989.
[4] D.D. Bainov, A.B. Dishliev, Uniform stability with respect to the impulse hypersurfaces of the solutions of differential equations with impulses, Internat. J. Systems Sci. 21 (1990) 2637–2643.
[5] X. Fu, X. Liu, Uniform boundedness and stability criteria in terms of two measures for impulsive differential equations, Appl. Math. Comput. 102 (1999) 237–255.
[6] S.I. Gurgula, Investigation of the stability of solutions of impulsive systems by Lyapunov's second method, Ukrainian Math. J. 1 (1982) 100–103 (in Russian).
[7] A. Halanay, D. Wexler, Qualitative Theory of Impulsive Systems, Editura Acad. RPR, Bucharest, 1968 (in Romanian).
[8] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[9] V. Lakshmikantham, X. Liu, On quasi stability for impulsive differential systems, Nonlinear Anal. 13 (1989) 819–828.

[10] V. Lakshmikantham, X. Liu, Impulsive hybrid systems and stability theory, Dynam. Contin. Discrete Impuls. Systems 7 (1998) 1–9.

[11] X. Liu, Quasi stability via Lyapunov functions for impulsive differential systems, Appl. Anal. 31 (1988) 201–213.

[12] V.D. Milman, A.D. Myshkis, On motion stability with shocks, Sibirsk. Mat. Zh. 1 (1960) 233–237 (in Russian).

[13] M. Rama Mohana Rao, S.K. Srivastava, Stability of Volterra integro-differential equations with impulse effect, J. Math. Anal. Appl. 163 (1992) 47–59.

[14] N. Rouche, P. Habets, M. Laloy, Stability Theory by Lyapunov's Direct Method, Springer-Verlag, New York, 1977.

[15] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.